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Reversing the drift of the Ehrenfest urn model and three conditionings

Thierry Huillet

Laboratoire de Physique Théorique et Modélisation, CNRS-UMR 8089 et Université de Cergy-Pontoise, 2 Avenue Adolphe Chauvin, F-95302, Cergy-Pontoise, France

E-mail: Thierry.Huillet@u-cergy.fr

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Abstract

We consider a drift-reversed version of the celebrated Ehrenfest urn process with N balls. For this 'dual' process, the boundaries are assumed to be absorbing and so the killing times at the boundaries play a central role. Three natural conditionings on the fixation/extinction events pertaining to this model are investigated. Some spectral information on the conditioned Markoff chains is obtained, allowing us to draw precise new conclusions on their limiting behaviors.

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1. Introduction and outline of the results

The Ehrenfest urn model (alias 'dogs and fleas') is a famous physical ball-in-box model for the heat exchange between two bodies (urns) at unequal temperature (the number of balls labeled 1 to N within each urn); see [7]. It was designed to elucidate some paradoxes raised by Boltzmann while considering the kinetic theory of gases, among which is irreversibility versus recurrence of states; see [18] for a review. The relevant state X_n counts the number of balls within say urn 1 after n steps, one interchanging step consisting in selecting at random a number within $\{1, \ldots, N\}$, determining the urn where the ball with this number is and moving it to the other urn. As is well known, X_n is a standard positive recurrent, 2-periodic birth-and-death (BD) Markoff chain with state space $\{0, \ldots, N\}$. This simple model is an exactly solvable Markoff chain because its spectral representation was completely determined by Kac and Krawtchouk, among others. For instance, the composition of urn 1 probability at each time is exactly known; see [8, 10, 13, 15]. The mathematician Mark Kac wrote about this model that it was: '. . . probably one of the more instructive models in the whole of Physics . . . ', [14].

In this paper, we shall be concerned with a related ball-in-box model which is 'dual' to that of Ehrenfest in the sense that the chain local probabilities of moving up and down are switched in the bulk of the state space, which amounts to reversing the drift of X_n; see [5], where this type of duality was coined the 'Wall duality', although there it concerns BD chains evolving on a semi-infinite state space with a reflecting barrier. For our drift-reversed model, the boundaries $\partial := \{0, N\}$ will be assumed to be absorbing. In terms of stochastics, this problem may be recast as a new BD Markoff chain, say $\widehat{X}_n$, still with state space $\{0, \dots, N\}$ but which is being absorbed at ∂. We shall, therefore, let $\widehat{\tau}_x := \inf(n : \widehat{X}_n \in \partial \text{ given } \widehat{X}_0 = x)$ be the first time ∂ is hit when started at x. This model was first considered in [11] from a purely combinatorial point of view and was called the 'Mabinogion urn model' for reasons explained therein. The physical motivations were to design a simple model of the spread of influences amongst versatile populations and draw some conclusions about the hitting times of the boundaries for various initial conditions of interest, together with some limit laws as N gets large. Again, see [11] for additional physical motivations.

We wish to shed some new light on this dual process by focusing on conditioning $\widehat{X}_n$ on the fixation/extinction events at the boundaries. We refer to [2, 12] for similar concerns in population genetics.

We shall, therefore, investigate the three following conditional problems pertaining to $\widehat{X}_n$ and of general interest.

(i) Consider the drift-reversed urn process $\widehat{X}_n$, started at $\widehat{X}_0 = x \in \{1, \dots, N-1\}$. Recall $\widehat{\tau}_x$ is its absolute extinction time at ∂. Then, conditional on the event $(\widehat{\tau}_x > n)$, $\widehat{X}_n$ turns out to be a positive recurrent Markov chain on $\{1, \dots, N-1\}$ whose Yaglom invariant (stationary) measure will be shown to be uniform. This measure is also the quasi-stationary distribution (qsd) of the chain. We refer to [4] for an introduction to similar concerns in the same framework.

(ii) A more stringent conditioning is also investigated: assuming the chain $\widehat{X}_n$ given $\widehat{X}_0 = x$, to be conditioned on never getting killed in the distant future, this conditional process is also a positive recurrent Markov chain on $\{1, \dots, N-1\}$ but whose invariant measure will now be shown to be binomial $\text{bin}(N-2, 1/2)$. Curiously, its probability transition matrix turns out to be that of a reversible Ehrenfest urn process but with $N-2$ balls.

Both results make use of a simple duality result displayed in lemma 1. This property makes the invariant laws of the conditioned processes explicit which is a rare event in the context of qsds for finite BD chains; see [4, 17].

(iii) Finally, we shall consider the drift-reversed random walk (RW) $\widehat{X}_n$ conditioned on exit through state $\{N\}$ using an appropriate Doob transform built on the harmonic (or scale) function of the chain. We shall use this transformation to give some information on the time it takes for this conditioned RW to undergo a sweep, that is to move from one endpoint of the state space to the other opposite absorbing state. Our results constitute a drift-reversed refined version of a result displayed in [1] for the usual Ehrenfest model.

2. Preliminary information: drift reversal of the Ehrenfest urn model

The Ehrenfest urn model is a BD process on $\{0, \dots, N\}$, say X_n, whose up and down local transition probabilities $X_n \to X_n \pm 1$ given $X_n = x$ are given by $p_x = 1 - x/N, x = 0, \dots, N-1, q_x = x/N, x = 1, \dots, N$. Its one-step transition matrix, therefore, is of the Jacobi type (empty entries are 0):

$$P = \frac{1}{N}\begin{bmatrix} 0 & N & & & & \\ 1 & 0 & N-1 & & & \\ & 2 & 0 & N-2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & N-1 & 0 & 1 \\ & & & & N & 0 \end{bmatrix}.$$

The set of the eigenvalues of P has been known for long to be $\lambda_k = 1 - \frac{2k}{N}, k = 0, \ldots, N$; see [13], for example. The Ehrenfest urn process is well known to be positive recurrent and 2-periodic with symmetric binomial bin$(N, 1/2)$ invariant measure.

By drift reversal (Wall duality), let us exchange p_x and q_x within the state-space bulk $\{1, \ldots, N-1\}$ and force the states 0 and N to be absorbing. The reversed chain, say $\widehat{X}_n$, is a new BD chain with transition probabilities $\widehat{p}_x = q_x, \widehat{q}_x = p_x, x = 1, \ldots, N-1$. We obtain the stochastic transition matrix of the Mabinogion urn model with dimensions $(N+1)\times(N+1)$:

$$\widehat{P} = \frac{1}{N}\begin{bmatrix} N & 0 & & & & \\ N-1 & 0 & 1 & & & \\ & N-2 & 0 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & N-1 \\ & & & & 0 & N \end{bmatrix}.$$

The drift of the RW $\widehat{X}_n$ is thus

$$\mathbb{E}_{\widehat{X}_n=x}(\widehat{X}_{n+1} - \widehat{X}_n) = \widehat{p}_x - \widehat{q}_x = -(p_x - q_x) = -\mathbb{E}_{X_n=x}(X_{n+1} - X_n),$$

but the conditional local fluctuations,

$$\sigma^2_{\widehat{X}_n=x}(\widehat{X}_{n+1} - \widehat{X}_n) = 4\widehat{p}_x\widehat{q}_x = 4p_xq_x = \sigma^2_{X_n=x}(X_{n+1} - X_n),$$

are left unchanged.

Let us erase both the first and last lines and columns of $\widehat{P}$. We obtain the matrix of reduced size $(N-1)\times(N-1)$:

$$\widetilde{P} = \frac{1}{N}\begin{bmatrix} 0 & 1 & & & & \\ N-2 & 0 & 2 & & & \\ & N-3 & 0 & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & 0 & N-2 \\ & & & & 1 & 0 \end{bmatrix}.$$

Let $\mathbf{1} := (1, 1, \ldots, 1)'$ be a column vector of 1 s with $'$ standing for transposition. The matrix $\widetilde{P}$ is substochastic with

$$\widetilde{P}\mathbf{1} = \frac{1}{N}(1, N, N, \ldots, N, 1)', \tag{1}$$

therefore it loses mass through the states $\{1\}$ and $\{N-1\}$. Let us now emphasize the following obvious but remarkable fact:

Lemma 1. *With $\widetilde{P}'$ the transpose of $\widetilde{P}$, it holds that*

$$\frac{N}{N-2}\widetilde{P}' = \frac{1}{N-2}\begin{bmatrix} 0 & N-2 & & & & \\ 1 & 0 & N-3 & & & \\ & 2 & 0 & N-4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & N-3 & 0 & 1 \\ & & & & N-2 & 0 \end{bmatrix},$$

so that $\frac{N}{N-2}\widetilde{P}'$ is again an Ehrenfest transition matrix but with $N-2$ balls.

From this result, we may draw the following spectral conclusions that will be used in the following. The spectrum of $\frac{N}{N-2}\widetilde{P}'$ is $1-2(k-1)/(N-2), k=1,\ldots,N-1$. That of $\widetilde{P}'$, and therefore of $\widetilde{P}$, are

$$\lambda_k = \frac{N-2}{N}\left(1-2\frac{k-1}{N-2}\right) = 1-\frac{2k}{N}, \qquad k=1,\ldots,N-1.$$

The spectrum of $\widehat{P}$ is $1-\frac{2k}{N}, k=1,\ldots,N-1$, to which one should add the pair $\{1,1\}$ corresponding to the two absorbing states $\{0,N\}$. The dominant eigenvalue of $\widehat{P}$ just after the double eigenvalue $\{1\}$ therefore is $1-2/N$.

3. Drift reversal and quasi-stationary distribution

Let $\widehat{P}$ be the $(N+1)\times(N+1)$ forward Jacobi transition matrix of the Mabinogion urn process. For such a discrete Markov chain, say $\widehat{X}_n$, started at $\widehat{X}_0 = x$, the hitting time $\widehat{\tau}_x$ of the absorbing boundaries $\{0,N\}$ is finite with probability 1. We have $\widehat{\tau}_x = \widehat{\tau}_{x,0}\wedge\widehat{\tau}_{x,N}$, the infimum of the hitting times of states $\{0\}$ and $\{N\}$, starting from x.

Let $\widetilde{P}$ be the $(N-1)\times(N-1)$ sub-matrix of $\widehat{P}$ obtained by removing the states $\{0,N\}$ from the entries of $\widehat{P}$. It is sub-stochastic at states $\{1,N-1\}$ with entries satisfying (1). The matrix $\widetilde{P}$ is easily seen to be the transition matrix of the process $\widehat{X}_n\cdot\mathbf{1}(\widehat{\tau}_x>n)$ (that is $\widehat{X}_n$, restricted to the set $\widehat{\tau}_x>n$). In other words,

$$\widetilde{P}^n(x,y) = \mathbb{P}_x(\widehat{X}_n = y; \widehat{\tau}_x > n). \tag{2}$$

Let $\rho<1$ be the spectral radius of $\widetilde{P}$, with from lemma 1 : $\rho = 1-2/N$. The matrix $\widetilde{P}$ is irreducible with period 2 whereas $\widetilde{P}^2$ is irreducible and aperiodic, with spectral radius ρ^2. By Perron–Frobenius theorem for primitive non-negative matrices

$$\lim_{n\uparrow\infty}\rho^{-2n}\widetilde{P}^{2n} = \psi\mu', \tag{3}$$

where $\mu'>\mathbf{0}$ ($\psi>\mathbf{0}$) is the left- (right-) line (column) eigenvector of $\widetilde{P}$ associated with ρ, defined by $\mu'\widetilde{P} = \rho\mu'$ and $\widetilde{P}\psi = \rho\psi$. Let us fix the normalization of the Perron vectors as follows: we will choose μ so that it constitutes a probability distribution vector so with $\sum_{x=1}^{N-1}\mu(x) = 1$. We will also choose ψ so that $\mu'\psi = 1$. It results from (2, 3) that

$$\lim_{n\uparrow\infty}\rho^{-2n}\mathbb{P}_x(\widehat{\tau}_x > 2n) = \psi(x), \tag{4}$$

so that $\widehat{\tau}_x$ has geometric tails with exponent ρ: on the set $(\widehat{\tau}_x>2n)$, the Markov chain $\widehat{X}_{2n}$ with values in $\{1,\ldots,N-1\}$ is called geometrically transient. The latter formula gives the limiting interpretation of ψ in our context. We also note from (2, 3) that

$$\mathbb{P}_\mu(\widehat{\tau} > 2n) := \sum_{x=1}^{N-1}\mu(x)\mathbb{P}_x(\widehat{\tau}_x > 2n) = \rho^{2n}. \tag{5}$$

If the process is started with μ, the law of the hitting time $\widehat{\tau}$ of ∂ is *exactly* geometric (ρ) distributed on $\{2, 4, \ldots\}$.

Consider now the conditional probability $\mathbb{P}_x(\widehat{X}_{2n} = y \mid \widehat{\tau}_x > 2n)$. One can easily check from (2)–(4) that, independently of the starting point x,

$$\mathbb{P}_x(\widehat{X}_{2n} = y \mid \widehat{\tau}_x > 2n) \underset{n\uparrow\infty}{\to} \mu(y).$$

As such, the probability measure μ can be interpreted as a Yaglom limit of $\widehat{X}_n$; see [19]. Furthermore, for each n

$$\mathbb{P}_\mu(\widehat{X}_{2n} = y \mid \widehat{\tau} > n) = \mu(y), y \in \{1, 2, \ldots, N-1\},$$

and μ is the (unique) qsd of $\widehat{X}_n$. As is well known for Markov chains with finite state space, the Yaglom limit coincides with the qsd; see [3, 17] for qsd examples in much more general contexts.

Applying these general ideas to the Mabinogion urn model, one obtains

Theorem 1.

(i) For the Mabinogion urn model, the Yaglom limit of the process conditioned on not yet being killed at the current time is the uniform distribution on the set $\{1, \ldots, N-1\}$:

$$\mu' = (1/(N-1), \ldots, 1/(N-1)). \tag{6}$$

(ii) With $\widehat{\tau}_x = \widehat{\tau}_{x,0} \wedge \widehat{\tau}_{x,N}$, *for each* $x \in \{1, \ldots, N-1\}$, *we have*

$$\lim_{n\uparrow\infty} (1-2/N)^{-2n} \mathbb{P}_x(\widehat{\tau}_x > 2n) = (N-1) \cdot 2^{-(N-2)} \binom{N-2}{x-1}, \tag{7}$$

and, with μ *uniform on* $\{1, \ldots, N-1\}$,

$$\mathbb{P}_\mu(\widehat{\tau} > 2n) = (1-2/N)^{2n}. \tag{8}$$

Proof. In our case study, $\rho = 1 - 2/N$ (the third largest eigenvalue after $\lambda_0 = \lambda_1 = 1$ for $\widehat{P}$ which is also the dominant eigenvalue of $\widetilde{P}$) and $\mu = 1/(N-1) \cdot \mathbf{1}$ can be checked to satisfy $\mu'\widetilde{P} = \left(1 - \frac{2}{N}\right)\mu'$, and so the Yaglom limit is the uniform distribution. Finally, the right eigenvector ψ associated with $\widetilde{P}$ and $\rho = 1 - 2/N$ can easily be checked to be $\psi(x) = K \cdot \binom{N-2}{x-1}$, $x = 1, \ldots, N-1$, for some constant $K = (N-1) \cdot 2^{-(N-2)}$ consistently with $\mu'\psi = 1$. □

4. Conditioning on no-exit in the remote future

Consider now the proper Markov chain whose transition probability matrix $\widetilde{\Pi}$ is obtained from $\widetilde{P}$ by the Doob transform:

$$\widetilde{\Pi}(x, y) = \rho^{-1} \frac{\psi(y)}{\psi(x)} \widetilde{P}(x, y), \qquad x, y \in \{1, \ldots, N-1\},$$

satisfying $\widetilde{\Pi}\mathbf{1} = \mathbf{1}$. In the matrix form, this transformation reads $\widetilde{\Pi} = \rho^{-1} D_\psi^{-1} \widetilde{P} D_\psi$.

The invariant probability distribution $\widetilde{\pi}$ on $\{1, \ldots, N-1\}$ satisfying $\widetilde{\pi}'\widetilde{\Pi} = \widetilde{\pi}'$ exists. Clearly, it is given explicitly by

$$\widetilde{\pi}(x) = \psi(x)\mu(x), \qquad x = 1, \ldots, N-1, \tag{9}$$

where μ and ψ are defined in (3). The Markov chain with the transformed transition probability matrix $\widetilde{\Pi}$ is that of the process whose n-step probability distribution is

$$\lim_{n'\uparrow\infty} \mathbb{P}_x(\widehat{X}_n = y \mid \widehat{\tau}_x > n'). \tag{10}$$

It corresponds to $\widehat{X}_n$ conditioned never to hit the coffin state ∂ in the distant future. This process has a unique invariant measure given by $\widetilde{\pi}$ in (9). This second conditioning being more stringent than the previous one, one should intuitively charge more heavily the sample paths staying away from $\{0, N\}$. We, indeed, have

Theorem 2.

(i) *For the Mabinogion urn model, the invariant measure of the corresponding process conditioned on never getting killed in the remote future is binomial bin$(N-2, 1/2)$, namely,*

$$\widetilde{\pi}(x) = 2^{-(N-2)}\binom{N-2}{x-1}, \qquad x = 1, \ldots, N-1. \tag{11}$$

(ii) *For the Mabinogion urn model with N balls, the transition matrix of the corresponding conditioned process is that of the Ehrenfest urn model with $N-2$ balls.*

Proof.

(i) The limiting qsd μ was shown to be uniform, and the right eigenvector ψ associated with $\widetilde{P}$ and $\rho = 1 - 2/N$ was checked to be $\psi(x) = K \cdot \binom{N-2}{x-1}$, $x = 1, \ldots, N-1$, where $K = (N-1) \cdot 2^{-(N-2)}$. Equation (11) follows from (9). Note that, in contrast to the uniform Yaglom limit law μ, the probability mass of $\widetilde{\pi}$ is more concentrated on the central atoms close to $N/2$, away from $\{0, N\}$.

(ii) It follows from $\widetilde{\Pi} = \rho^{-1} D_{\psi}^{-1} \widetilde{P} D_{\psi}$ and the expressions of $\psi, \widetilde{P}$ and ρ. For instance, with $\widetilde{P}(x, x+1) = x/N$,

$$\widetilde{\Pi}(x, x+1) = \frac{N}{N-2}\binom{N-2}{x-1}^{-1}\frac{x}{N}\binom{N-2}{x} = 1 - \frac{x}{N-2},$$

which is the transition probability, $x \to x+1$, of the Ehrenfest model with $N-2$ balls.

□

5. Conditioning on exit in state $\{N\}$

Consider again the reversed BD chain, $\widehat{X}_n$ with the transition matrix $\widehat{P}$ where both states $\{0, N\}$ are absorbing. Define the scale (or harmonic) function φ of this BD RW as the function which makes $M_n := \varphi\left(\widehat{X}_{n\wedge\tau_{x,0}}\right)$ a martingale. The function φ is important because, as is well known, for all $0 < x < N$, with $\widehat{\tau}_x = \widehat{\tau}_{x,0} \wedge \widehat{\tau}_{x,N}$ the first hitting time of $\{0, N\}$ starting from x,

$$\mathbb{P}_x\left(\widehat{X}_{\widehat{\tau}_x} = N\right) = \mathbb{P}(\widehat{\tau}_{x,N} < \widehat{\tau}_{x,0}) = \frac{\varphi(x)}{\varphi(N)}, \tag{12}$$

gives a closed-form expression of the probability of exit at $\{N\}$ before hitting $\{0\}$. The searched harmonic function is $\varphi(x) =: 1 + \sum_{y=1}^{x-1}\psi(y)$, where $\psi(y)$ satisfies $\widehat{q}_y\psi(y-1) = \widehat{p}_y\psi(y)$, with $\psi(1) := 1$. Thus $\psi(y) = \prod_{z=1}^{y}\frac{\widehat{q}_z}{\widehat{p}_z}$ and so, with $x = 1, \ldots, N$, $\varphi(0) := 0$:

$$\varphi(x) = 1 + \sum_{y=1}^{x-1}\prod_{z=1}^{y}\frac{\widehat{q}_z}{\widehat{p}_z} = 1 + \sum_{y=1}^{x-1}\prod_{z=1}^{y}\frac{p_z}{q_z} = 1 + \sum_{y=1}^{x-1}\binom{N-1}{y} \tag{13}$$

is a cumulative binomial sequence. Note $\varphi(1) = 1$. Let us consider the problem of conditioning $\widehat{X}_n$ on exiting in state $\{N\}$. Let $D_{\varphi} := \mathrm{diag}(\varphi(0), \ldots, \varphi(N))$. The transition matrix $\widehat{Q}_c$ of this conditioned RW is given in terms of the Doob transform [6]:

$$\widehat{Q}_c = D_{\varphi}^{-1}\widehat{P}D_{\varphi}. \tag{14}$$

As a result, the exact shape of $\widehat{Q}_c$ is

$$\widehat{Q}_c = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 0 & \frac{\varphi(2)}{\varphi(1)}\frac{1}{N} = 1 & & & \\ 0 & \frac{\varphi(1)}{\varphi(2)}\left(1-\frac{2}{N}\right) & 0 & \frac{\varphi(3)}{\varphi(2)}\frac{2}{N} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{\varphi(N-2)}{\varphi(N-1)}\frac{1}{N} & 0 & \frac{\varphi(N)}{\varphi(N-1)}\left(1-\frac{1}{N}\right) \\ & & & & 0 & 1 \end{bmatrix},$$

where it is apparent that state $\{0\}$ became isolated and disconnected in the transformation process ($\widehat{Q}_c(0,0) = 1$ and $\widehat{Q}_c(0,1) = \widehat{Q}_c(1,0) = 0$). Deleting the line and the row $\{0\}$ of $\widehat{Q}_c$, we get a stochastic matrix, call it $\widehat{P}_c$, of a process say $\widehat{X}_n^c$ now on the state space $\{1,\ldots,N\}$ which corresponds to $\widehat{X}_n$ conditioned to first hit state $\{N\}$ before state $\{0\}$. As required, if state $\{0\}$ is to be inaccessible after conditioning, then state $\{1\}$ of this conditioned BD process now is purely reflecting ($\widehat{Q}_c(1,2) = 1$), whereas state $\{N\}$ remains absorbing ($\widehat{Q}_c(N,N) = 1$). $\widehat{Q}_c$ and $\widehat{P}$ are diagonally similar and so both share the same spectrum, which is $\lambda_k = 1 - \frac{2k}{N}, k = 1,\ldots,N-1$ to which one should add the eigenvalues $\{1,1\}$. Putting this together, we get

Proposition 1.

(i) The transition matrix of the Mabinogion urn model conditioned on exit at $\{N\}$ is the restriction $\widehat{P}_c$ of $\widehat{Q}_c$ to $\{1,\ldots,N\}$, where the scale function φ is given by (13).

(ii) The spectrum of $\widehat{P}_c$ is reduced to $\{1\} \cup \left\{1 - \frac{2k}{N}, k = 1,\ldots,N-1\right\}$.

For this conditioned process, it is of importance to evaluate the time $\widehat{\tau}_{1,N}^c$ it takes to reach $\{N\}$ starting from state $\{1\}$, joining the two extreme states of this process. We have

Theorem 3.

(i) The probability generating function of $\widehat{\tau}_{1,N}^c \geqslant N-1$ is

$$\mathbb{E}\left(z^{\widehat{\tau}_{1,N}^c}\right) = \prod_{k=1}^{N-1} \frac{(1-\lambda_k)z}{1-\lambda_k z}, \qquad z \in [0,1], \tag{15}$$

where $-1 < \lambda_k = 1 - \frac{2k}{N} < +1, k = 1,\ldots,N-1$ are the $N-1$ distinct eigenvalues of $\widehat{P}_c$, avoiding $\lambda_0 = 1$.

(ii) In particular,

$$\mathbb{E}\left(\widehat{\tau}_{1,N}^c\right) \sim_{N\uparrow\infty} \frac{N}{2}\log N \qquad \textit{and} \qquad \sigma^2\left(\widehat{\tau}_{1,N}^c\right) \sim_{N\uparrow\infty} \left(\frac{N}{2}\right)^2. \tag{16}$$

(iii)

$$\frac{\widehat{\tau}_{1,N}^c}{\mathbb{E}\left(\widehat{\tau}_{1,N}^c\right)} \underset{N\uparrow\infty}{\to} 1 \quad \textit{in probability.} \tag{17}$$

Proof.

(i) Equation (15) is a particular incarnation of an algebraic result described in [16, section 1] and [9, theorem 1.2] in the context of BD chains absorbed at $\{N\}$. In some cases, there is a probabilistic proof; see [9, theorem 4.2]. This part makes use of our findings on the spectrum of $\widehat{P}_c$ which makes (15) explicit.

(ii) From (i), we have $\mathbb{E}\left(\widehat{\tau}_{1,N}^{c}\right) = \sum_{k=1}^{N-1} \frac{1}{1-\lambda_k}$ and

$$\sigma^2\left(\widehat{\tau}_{1,N}^{c}\right) = \sum_{k=1}^{N-1}(1-\lambda_k)^{-2} - \sum_{k=1}^{N-1}(1-\lambda_k)^{-1}.$$

Because the eigenvalues λ_k are known leading to $1-\lambda_k = \frac{2k}{N}$, using the integral approximation

$$\mathbb{E}\left(\widehat{\tau}_{1,N}^{c}\right) \sim N\int_0^1 \frac{1}{2}\frac{\mathrm{d}x}{(x+1/N)},$$

we easily get

$$\mathbb{E}\left(\widehat{\tau}_{1,N}^{c}\right) \sim \frac{N}{2}\log N \qquad \text{and} \qquad \sigma^2\left(\widehat{\tau}_{1,N}^{c}\right) \sim \left(\frac{N}{2}\right)^2,$$

showing that $\sigma^2(\widetilde{\tau}_{0,N}/\mathbb{E}(\widetilde{\tau}_{0,N})) \sim (\log N)^{-2} \to 0$ as $N \uparrow \infty$. (iii) follows from this last statement. □

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